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b) Use the 2 equations from part (a) together with thermal-wind balance to show that the QG divergent circulation is determined diagnostically by

$$L(w) = 2f\Lambda \frac{\partial^2}{\partial x^2} v_g', \tag{1}$$

where *L* is an elliptic differential operator. Make a diagram to show the sense of this circulation in a region of cyclonic relative vorticity, $\zeta_g' > 0$. Interpret the right-hand side of (1) (the "forcing") in terms of the advection of heat and vorticity contained in either the perturbation or basic flow.

c) Show that if the forcing in (1) is localized, then a "downgradient" heat flux, *i.e.*,

$$\frac{d\bar{b}}{dy} \int\int v'b' dxdz < 0 \tag{2}$$

implies a thermally "direct" divergent circulation, whereas an upgradient flux is associated with an indirect cell. (Recall that a direct circulation has

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2. Coupled Ekman layers

The Ekman equations for the planetary boundary layer are obtained by writing the *f*-plane momentum equation as

$$d\boldsymbol{V}/dt = \ldots - \rho^{-1} d\tau/dz, \tag{4}$$

with $\tau = -\rho K \partial \boldsymbol{V}/\partial z$, and neglecting the left-hand side. An *independent* formula often used for the stress at the ground is $\tau = -\rho C \boldsymbol{V}$ at $z = 0$. Here C is a constant with dimensions of velocity.

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a) Obtain the simplest form of the quasi-geostrophic forecast equations for the vorticity and buoyancy, ζ_g' and b', in terms of the vertical velocity w and the constants f, Λ and N^2. Take full advantage of the assumed symmetry of the perturbation, which gives not only $\partial/\partial y = 0$ for the perturbation, but also $u_g' = 0$ (please tell why).

b) Use the 2 equations from part (a) together with thermal-wind balance to show that the QG divergent circulation is determined diagnostically by

$$L(w) = 2f\Lambda\frac{\partial^2}{\partial x^2}v_g', \tag{1}$$

where L is an elliptic differential operator. Make a diagram to show the sense of this circulation in a region of cyclonic relative vorticity, $\zeta_g' > 0$. Interpret the right-hand side of (1) (the "forcing") in terms of the advection of heat and vorticity contained in either the perturbation or basic flow.

c) Show that if the forcing in (1) is localized, then a "downgradient" heat flux, *i.e.*,

$$\frac{d\bar{b}}{dy}\iint v'b'dxdz < 0 \tag{2}$$

implies a thermally "direct" divergent circulation, whereas an upgradient flux is associated with an indirect cell. (Recall that a direct circulation has

$$\iint w'b'dxdz > 0 \tag{3}$$

by definition.)

2. Coupled Ekman layers

The Ekman equations for the planetary boundary layer are obtained by writing the *f*-plane momentum equation as

$$dV/dt = \ldots - \rho^{-1} d\tau/dz , \tag{4}$$

with $\tau = -\rho K \partial V/\partial z$, and neglecting the left-hand side. An *independent* formula often used for the stress at the ground is $\tau = -\rho C V$ at $z = 0$. Here C is a constant with dimensions of velocity.

a) Using the above model and assuming a geostrophic zonal wind, $V_g = U\hat{x}$, in the free atmosphere, find an expression for the wind vector at the ground. Take ρ and K constant. Discuss the limits $C = 0$ and $C = \infty$. Be careful not to incorporate the traditional assumption that the wind is zero at the ground!

b) Find the vertically integrated horizontal mass flux due to this boundary layer.

c) When the stress on the atmosphere is τ, the stress on the underlying ocean is obviously $-\tau$. Determine the direction and strength of the surface current in terms of the surface wind, the oceanic Ekman depth and the ratio ρ/ρ_o, where ρ_o is the ocean water density. How does the integrated boundary-layer mass flux in the ocean compare with that in the atmosphere?

3. Rossby adjustment problem for vortex strip

Consider the shallow-water equations linearized about a state of rest with mean depth H. As an initial condition, we have a meridional velocity v' consisting of a shear flow that is a linear function of x alone, in three pieces:

$$v' = \begin{cases} -\tilde{\zeta}L, & x < -L \\ \tilde{\zeta}x, & |x| \le L \\ \tilde{\zeta}L, & L < x \end{cases} \tag{5}$$

where $\tilde{\zeta}$ is the vorticity of the vortex strip between $-L$ and L. The initial condition on the height is $h' = 0$.

a) What is the equilibrium state of the flow for the *non-rotating* case? What is the initial and final kinetic and potential energy. **Hint: This problem is tricky but simple: show that this initial condition is a valid solution of the equations of motion for all time, and, hence, at equilibrium. Remember that there is no y dependence.**

b) What is the equilibrium state of the flow for the *rotating* case, with a constant Coriolis parameter f? Should the solution be symmetric or antisymmetric about $x = 0$? Use this fact to simplify the problem. Sketch the velocity and height at equilibrium. Compare the energy in this case to that of case a).

4. Edge waves on a circular vortex patch: Consider an *axisymmetric* barotropic shear flow specified in polar coordinates by

$$\zeta = \begin{cases} 2\Omega, & r \le R \\ 0, & r > R \end{cases} \qquad (6)$$

for constants Ω and R.

(a) Assuming that the azimuthal velocity V matches at $r = R$, find V as a function of r.

(b) Perturb the boundary of the circular region of constant vorticity with a small-amplitude sinusoidal disturbance. Find the phase speed of this disturbance in terms of its azimuthal wave-number. Show how this result reduces to the dispersion relation 6.50 of the class notes as either the wavenumber or the radius R increases. **Hints: Review the notes, section 6.9, before attempting this problem. You will need to solve Laplace's equation in polar coordinates.**